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ENGINEERING MECHANICS

CHAPTER 8: MOTION

Lecture 2:

Velocity and acceleration by differentiation:

Sometimes, the given equation of motion is in terms of displacement (s) and time (t)

e.g., s = $3t^3$+ $2t^2$+ 6t + 4 ...(i)

or s = 6 + $5t^2$ + $6t^3$ ...(ii)

or s = $2t^3$ + 4t – 15 ...(iii)

Now differentiating both sides, of the equations, with respect to *t*,

$$\frac{ds}{dt} = 9t^2 + 4t + 6 \quad ...(i)$$

$$\frac{ds}{dt} = 10t + 18t^2 \quad ...(ii)$$

$$\frac{ds}{dt} = 6t^2 + 4 \quad ...(iii)$$

The equations, so obtained by differentiation, give velocity of the body (as the velocity of a body is the rate of change of its position). Again differentiating, both sides of the above equations, with respect to *t*,

$$\frac{d^2s}{dt^2} = 18t + 4 \quad ...(i)$$

$$\frac{d^2s}{dt^2} = 10 + 36t \quad ...(ii)$$

$$\frac{d^2s}{dt^2} = 12t \quad ...(iii)$$

The equations, so obtained by second differentiation, give acceleration of the body (as the acceleration of a body is the rate of change of velocity).

Example 18.1. *A particle, starting from rest, moves in a straight line, whose equation of motion is given by :* $s = t^3 - 2t^2 + 3$. *Find the velocity and acceleration of the particle after 5 seconds.*

Solution. Given : Equation of displacement : $s = t^3 - 2t^2 + 3$ ...(*i*)

Velocity after 5 seconds

Differentiating the above equation with respect to t,

$$\frac{ds}{dt} = 3t^2 - 4t \qquad ...(ii)$$

i.e., velocity, $v = 3t^2 - 4t$ $\qquad ...\left(\because \frac{ds}{dt} = \text{Velocity}\right)$

substituting t equal to 5 in the above equation,

$$v = 3\,(5)^2 - (4 \times 5) = 55 \text{ m/s} \qquad \textbf{Ans.}$$

Acceleration after 5 seconds

Again differentiating equation (*ii*) with respect to t,

$$\frac{d^2s}{dt^2} = 6t - 4 \qquad ...(iii)$$

i.e. acceleration, $a = 6t - 4$ $\qquad ...\left(\because \frac{d^2s}{dt^2} = \text{Acceleration}\right)$

Now substituting t equal to 5 in the above equation,

$$a = (6 \times 5) - 4 = 26 \text{ m/s}^2 \qquad \textbf{Ans.}$$

Example 18.2. *A car moves along a straight line whose equation of motion is given by* $s = 12t + 3t^2 - 2t^3$, *where (s) is in metres and (t) is in seconds. calculate*

(*i*) *velocity and acceleration at start, and*

(*ii*) *acceleration, when the velocity is zero.*

Solution. Given : Equation of displacement : $s = 12t + 3t^2 - 2t^3$ ...(*i*)

Velocity at start

Differentiating the above equation with respect to *t*,

$$\frac{ds}{dt} = 12 + 6t - 6t^2 \quad ...(ii)$$

i.e. velocity, $v = 12 + 6t - 6t^2$ $\quad ...\left(\because \frac{ds}{dt} = v\right)$

Substituting *t* equal to 0 in the above equation,

$v = 12 + 0 - 0 = 12$ m/s **Ans.**

Acceleration at start

Again differentiating equation (*ii*) with respect to *t* ,

$$\frac{dv}{dt} = 6 - 12t \quad ...(iii)$$

i.e. accleration, $a = 6 - 12t$ $\quad ...\left(\because \frac{dv}{dt} = a\right)$

Now substituting *t* equal to 0 in the above equation,

$a = 6 - 0 = 6$ m/s^2 **Ans.**

Acceleration, when the velocity is zero

Substituting equation (*ii*) equal to zero

$12 + 6t - 6t^2 = 0$

$t^2 - t - 2 = 0$ ...(Dividing by – 6)

or $t = 2$ s

It means that velocity of the car after two seconds will be zero. Now substituting the value of *t* equal to 2 in equation (*iii*),

$a = 6 - (12 \times 2) = - 18$ m/s^2 **Ans.**

Velocity and Acceleration by Integration:

Sometimes, the given equation of motion is in terms of acceleration (a) and time (t)

e.g. $a = 4t^3 - 3t^2 + 5t + 6$ ...(i)

$$\frac{dv}{dt} = t^3 + 8t \qquad \text{...(ii)}$$

$$v\frac{dv}{ds} = 6 + 3t \qquad \text{...(iii)}$$

$$\frac{d^2 s}{dt^2} = 4t - 8t^2 \qquad \text{...(iv)}$$

Now integrating both sides of the above equations,

$$= \frac{t^4}{4} + \frac{8t^2}{2} + C_1 \qquad \text{...(i)}$$

$$= 6t + \frac{3t^2}{2} + C_1 \qquad \text{...(ii)}$$

$$= \frac{4t^2}{2} - \frac{8t^3}{3} + C_1 \qquad \text{...(iii)}$$

where C_1 is the first constant of integration. The equations, so obtained, give the velocity of the body. Again integrating both sides of the above equations.

$$= \frac{t^5}{20} + \frac{8t^3}{6} + C_1 t + C_2$$

$$= \frac{6t^2}{2} + \frac{3t^3}{6} C_1 t + C_2$$

$$= \frac{4t^3}{6} - \frac{8t^4}{12} + C_1 t + C_2$$

where C_2 is the second constant of integration. The equations, so obtained, give the displacement of the body.

Example 18.4 *The motion of a particle is given by :*

$$a = t^3 - 3t^2 + 5$$

where (a) is the acceleration in m/s² and (t) is the time in seconds. The velocity of the particle at t = 1 second is 6.25 m/sec and the displacement is 8.8 metres.

Calculate the displacement and velocity at t = 2 seconds.

Solution. Given : Equation of acceleration : $a = t^3 - 3t^2 + 5$

Rewriting the given equation,

or $$\frac{dv}{dt} = t^3 - 3t^2 + 5 \qquad \text{...}\left(\because\ a = \frac{dv}{dt}\right)$$

∴ $$dv = (t^3 - 3t^2 + 5)\, dt \qquad \text{...(i)}$$

Velocity at t = 2 seconds

Integrating both sides of equation (*i*),

$$v = \frac{t^4}{4} - \frac{3t^3}{3} + 5t + C_1$$

$$= \frac{t^4}{4} - t^3 + 5t + C_1 \qquad \text{...(ii)}$$

where C_1 is the first constant of integration. Substituting the values of $t = 1$ and $v = 6.25$ in equation (*ii*),

$$6.25 = \frac{1}{4} - 1 + 5 + C_1 = 4.25 + C_1$$

$$\therefore \quad C_1 = 6.25 - 4.25 = 2$$

Substituting this value of C_1 in equation (*ii*),

$$v = \frac{t^4}{4} - t^3 + 5t + 2 \quad \text{...(iii)}$$

Now for velocity of the particle, substituting the value of $t = 2$ in the above equation ,

$$v = \frac{(2)^4}{4} - (2)^3 + (5 \times 2) + 2 = 8 \text{ m/s} \quad \textbf{Ans.}$$

Displacement at t = 2 seconds

Rewriting equation (*iii*),

$$\frac{ds}{dt} = \frac{t^4}{4} - t^3 + 5t + 2 \quad \text{...}\left(\because v = \frac{ds}{dt}\right)$$

$$\therefore \quad ds = \left(\frac{t^4}{4} - t^3 + 5t + 2\right) dt \quad \text{...(iv)}$$

Integrating both sides of equation, (*iv*)

$$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} + 2t + C_2 \quad \text{...(v)}$$

where C_2 is the second constant of integration. Substituting the values of $t = 1$ and $s = 8.8$ in equation (*v*),

$$8.8 = \frac{1}{20} - \frac{1}{4} + \frac{5}{2} + 2 + C_2 = 4.3 + C_2$$

$$\therefore \quad C_2 = 8.8 - 4.3 = 4.5$$

Substituting this value of C_2 in equation (*v*) ,

$$s = \frac{t^5}{20} - \frac{t^4}{4} + \frac{5t^2}{2} + 2t + 4.5$$

Now for displacement of the particle, substituting the value of $t = 2$ in the above equation,

$$s = \frac{32}{20} - \frac{16}{4} + \frac{20}{2} + 4 + 4.5 = 16.1 \text{ m} \quad \textbf{Ans.}$$

Example 18.5 *A train, starting from rest, is uniformly accelerated. The acceleration at any instant is* $\frac{10}{v+1}$ *m/s², where (v) is the velocity of the train in m/s at the instant. Find the distance, in which the train will attain a velocity of 35 km. p.h.*

Solution. Given : Equation of acceleration : $a = \frac{10}{v+1}$

Rewriting the given equation,

$$v \cdot \frac{dv}{ds} = \frac{10}{v+1} \qquad \ldots\left(\because a = v \cdot \frac{dv}{ds}\right)$$

$$\therefore \qquad v(v+1)\,dv = 10\,ds \qquad \ldots(i)$$

or $$(v^2 + v)\,dv = 10\,ds$$

Integrating both sides of equation (*i*),

$$\frac{v^3}{3} + \frac{v^2}{2} = 10s + C_1 \qquad \ldots(ii)$$

where C_1 is the first constant of integration. Substituting the values of $s = 0$ and $v = 0$ in equation (*ii*),

$$C_1 = 0$$

Substituting this value of $C_1 = 0$ in equation (*ii*),

$$\frac{v^3}{3} + \frac{v^2}{2} = 10s$$

$$\therefore \qquad 2v^3 + 3v^2 = 60\,s \qquad \ldots(iii)$$

Now for distance travelled by the train, substituting $v = 36$ km.p.h. or 10 m/s in equation (*iii*),

$$2\,(10)^3 + 3\,(10)^2 = 60\,s \text{ or } 2000 + 300 = 60\,s$$

$$s = \frac{2300}{60} = 38.3 \text{ m} \quad \textbf{Ans.}$$

OBJECTIVE TYPE QUESTIONS

1. We are given an equation of displacement (s) in terms of time (t). If we differentiate it with respect to t, the equation so obtaned will give
   (a) velocity (b) acceleration (c) distance traversed
2. The second differentiation, of the above equation will give
   (a) velocity (b) acceleration (c) distance traversed
3. If we differentiate an equation in terms of acceleration and time, it will give
   (a) velocity (b) distance traversed (c) none of these two
4. We are given an equation of acceleration (a) in terms of time (t). The second integration of the equation will give the velocity.
   (a) Yes (b) No
5. Which of the following statement is wrong ?
   (a) A body falling freely under the force of gravity is an exapmle of motion under variable acceleration.
   (b) A bus going down the valley may have variable acceleration.
   (c) A lift going down in a gold mine cannot have constant acceleration in the entire journey.
   (d) In a cricket match, the ball does not move with constant acceleration.

ANSWERS

1. (a) **2.** (b) **3.** (c) **4.** (b) **5.** (a)