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ENGINEERING MECHANICS

CHAPTER 8: MOTION

Lecture 3:

Rotational Motion: Rotational motion or motion of rotation (i.e., angular motion) takes place about the geometric axis of the body. The angular velocity of a body is always expressed in terms of revolutions described in one minute, e.g., if at an instant the angular velocity of rotating body in N r.p.m. (i.e. revolutions per min), the corresponding angular velocity ω (in rad) may be found out as discussed below :

$$\begin{aligned} 1 \text{ revolution/min} &= 2\pi \text{ rad/min} \\ \therefore \quad N \text{ revolutions/min} &= 2\pi N \text{ rad/min} \\ \text{and angular velocity} \quad \omega &= 2\pi N \text{ rad/min} \\ &= \frac{2\pi N}{60} \text{ rad/sec} \end{aligned}$$

Some Important terms:

1. Angular velocity: It is the rate of change of angular displacement of a body, and is expressed in r.p.m. (revolutions per minute) or in radian per second. It is, usually, denoted by ω (omega).

2. Angular acceleration: It is the rate of change of angular velocity and is expressed in radian per second per second (rad/s^2) and is usually, denoted by α. It may be constant or variable.

3. Angular displacement: It is the total angle, through which a body has rotated, and is usually denoted by θ. Mathematically, if a body is rotating with a uniform angular velocity (ω) then in t seconds, the angular displacement

$$\theta = \omega t$$

Motion of rotation under constant angular acceleration:

Consider a particle, rotating about its axis.

Let ω_0 = Initial angular velocity,

ω = Final angular velocity,

t = Time (in seconds) taken by the particle to change its velocity from ω_0 to ω.

α = Constant angular acceleration in rad/s^2, and

θ = Total angular displacement in radians.

Since in t seconds, the angular velocity of the particle has increased steadily from ω_0 to ω at the rate of α rad/s^2, therefore

$$\omega = \omega_0 + \alpha t \qquad ...(i)$$

and average angular velocity $= \dfrac{\omega_0 + \omega}{2}$

We know that the total angular displacement,

$$\theta = \text{Average velocity} \times \text{Time} = \left(\frac{\omega_0 + \omega}{2}\right) \times t \qquad ...(ii)$$

Substituting the value of ω from equation (i),

$$\theta = \frac{\omega_0 + (\omega_0 + \alpha t)}{2} \times t = \frac{2\omega_0 + \alpha t}{2} \times t = \omega_0 t + \frac{1}{2}\alpha t^2 \qquad ...(iii)$$

and from equation (i), we find that

$$t = \frac{\omega - \omega_0}{\alpha}$$

Substituting this value of t in equation (ii),

$$\theta = \left(\frac{\omega_0 + \omega}{2}\right) \times \left(\frac{\omega - \omega_0}{\alpha}\right) = \frac{\omega^2 - \omega_0^2}{2\alpha}$$

$$\therefore \quad \omega^2 = \omega_0^2 + 2\alpha\theta \qquad ...(iv)$$

Relation between linear motion and angular motion:

Following are the relations between the linear motion and the angular motion of a body :

S. No.	Particulars	Linear motion	Angular motion
1.	Initial velocity	u	ω_0
2.	Final velocity	v	ω
3.	Constant acceleration	a	α
4.	Total distance traversed	s	θ
5.	Formula for final velocity	$v = u + at$	$\omega = \omega_0 + \alpha t$
6.	Formula for distance traversed	$s = ut + \frac{1}{2}at^2$	$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$
7.	Formula for final velocity	$v^2 = u^2 + 2as$	$\omega^2 = \omega_0^2 + 2\alpha\theta$
8.	Differential formula for velocity	$v = \frac{ds}{dt}$	$\omega = \frac{d\theta}{dt}$
9.	Differential formula for acceleration	$a = \frac{dv}{dt}$	$\alpha = \frac{d\omega}{dt}$

Example. 21.1 *A flywheel starts from rest and revolves with an acceleration of 0.5 rad/sec^2. What will be its angular velocity and angular displacement after 10 seconds.*

Solution. Given : Initial angular velocity (ω_0) = 0 (becasue it starts from rest) ; Angular acceleration (α) = 0.5 rad/sec^2 and time (t) = 10 sec.

Angular velocity of the flywheel

We know that angular velocity of the flywheel,

$$\omega = \omega_0 + \alpha t = 0 + (0.5 \times 10) = 5 \text{ rad/sec} \quad \textbf{Ans.}$$

Angular displacement of the flywheel

We also know that angular displacement of the flywheel,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = (0 \times 10) + \left[\frac{1}{2} \times 0.5 \times (10)^2\right] = 25 \text{ rad} \quad \textbf{Ans.}$$

Example 21.2 *A wheel increases its speed from 45 r.p.m. to 90 r.p.m. in 30 seconds. Find (a) angular acceleration of the wheel, and (b) no. of revolutions made by the wheel in these 30 seconds.*

Solution. Given : Initial angular velocity (ω_0) = 45 r.p.m. = 1.5 π rad/sec ; Final angular velocity (ω) = 90 r.p.m. = 3 π rad/sec and time (t) = 30 sec

(a) Angular acceleration of the wheel

Let α = Angular acceleration of the wheel.

We know that final angular velocity of the wheel (ω),

$$3\pi = \omega_0 + \alpha t = 1.5\,\pi + (\alpha \times 30) = 1.5\pi + 30\alpha$$

or

$$\alpha = \frac{3\pi - 1.5\pi}{30} = \frac{1.5\pi}{30} = 0.05\pi \text{ rad/sec}^2 \quad \textbf{Ans.}$$

(*b*) *No. of revolutions made by the wheel in 30 seconds*

We also know that total angle turned by the wheel in 30 seconds,

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 = [1.5\pi \times 30] + \left[\frac{1}{2} \times 0.05\ \pi(30)^2\right] = 67.5\ \pi\ \text{rad}$$

$$= \frac{67.5\pi}{2\pi} = 33.75\,\text{rev} \quad \textbf{Ans.} \qquad \text{...(1 rev} = 2\pi\ \text{rad)}$$

Example 21.5. *A wheel rotates for 5 seconds with a constant angular acceleration and describes during this time 100 radians. It then rotates with a constant angular velocity and during the next five seconds describes 80 radians.*

Find the initial angular velocity and the angular acceleration.

Solution. Given : Time (t) = 5 sec and angular displacement (θ) = 100 rad

Initial angular velocity

Let ω_0 = Initial angular velocity in rad/s,

α = Angular acceleration in rad/s^2, and

ω = Angular velocity after 5 s in rad/s.

First of all, consider the angular motion of the wheel with constant acceleration for 5 seconds. We know that angular displacement (θ),

$$100 = \omega_0 t + \frac{1}{2}\alpha t^2 = \omega_0 \times 5 + \frac{1}{2} \times \alpha(5)^2 = 5\omega_0 + 12.5\alpha$$

$$\therefore \quad 40 = 2\omega_0 + 5\alpha \qquad \text{...}(i)$$

and final velocity, $\omega = \omega_0 + \alpha t = \omega_0 + \alpha \times 5 = \omega_0 + 5\alpha$

Now consider the angular motion of the wheel with a constant angular velocityof $(\omega_0 + 5\alpha)$ for 5 seconds and describe 80 radians. We know that the angular displacement,

$$80 = 5\,(\omega_0 + 5\alpha)$$

$$\text{or} \quad 16 = \omega_0 + 5\alpha \qquad \text{...}(ii)$$

Subtracting equation (*ii*) from (*i*),

$$24 = \omega_0 \quad \text{or} \quad \omega_0 = 24\ \text{rad/s} \quad \textbf{Ans.}$$

Angular acceleration

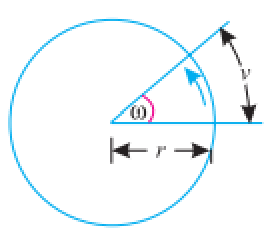
Substituting this value of ω_0 in equation (*ii*),

$$16 = 24 + 5\alpha \quad \text{or} \quad \alpha = \frac{16 - 24}{5} = -1.6\ \text{rad/s}^2 \quad \textbf{Ans.}$$

...(Minus sign means retardation)

Linear or tangential velocity of a rotating body:

We consider a body rotating about its axis as shown in figure:



Let ω = Angular velocity of the body in rad/s,

r = Radius of the circular path in metres, and

v = Linear velocity of the particle on the periphery in m/s.

After one second, the particle will move v metres along the circular path and the angular displacement will be ω rad.

We know that length of arc = Radius of arc × Angle subtended in rad.

$$\therefore \quad v = r\,\omega$$

Example 21.9. *A wheel of 1.2 m diameter starts from rest and is accelerated at the rate of 0.8 rad/s^2 . Find the linear velocity of a point on its periphery after 5 seconds.*

Solution. Given : Diameter of wheel = 1.2 m or radius (r) = 0.6 m ; Initial angular velocity (ω_0) = 0 (becasue, it starts from rest) ; Angular acceleration (α) = 0.8 rad/s^2 and time (t) = 5 s

We know that angular velocity of the wheel after 5 seconds,

$$\omega = \omega_0 + \alpha t = 0 + (0.8 \times 5) = 4 \text{ rad/s}$$

$\therefore$ Linear velocity of the point on the periphery of the wheel,

$$v = r\omega = 0.6 \times 4 = 2.4 \text{ m/s} \quad \textbf{Ans.}$$

Example 21.10. *A pulley 2 m in diameter is keyed to a shaft which makes 240 r.p.m. Find the linear velocity of a particle on the periphery of the pulley.*

Solution. Given : Diameter of pulley = 2 m or radius (r) = 1 m and angular frequency (N) = 240 r.p.m.

We know that angular velocity of the pulley,

$$\omega = \frac{2\pi N}{60} = \frac{2\pi \times 240}{60} = 25.1 \text{ rad/s} \quad \textbf{Ans.}$$

$\therefore$ Linear velocity of the particle on the periphery of the pulley,

$$v = r\omega = 1 \times 25.1 = 25.1 \text{ m/s} \quad \textbf{Ans.}$$

Linear or tangential acceleration of a rotating body:

Consider a body rotating about its axis with a constant angular (as well as linear) acceleration. We know that linear acceleration,

$$a = \frac{dv}{dt} = \frac{d}{dt}(v) \quad \quad ...(i)$$

We also know that in motion of rotation, the linear velocity,

$$v = r\omega$$

Now substituting the value of v in equation (i),

$$a = \frac{d}{dt}(r\omega) = r\frac{d\omega}{dt} = r\alpha$$

Where α = Angular acceleration in rad/sec^2 and is equal to $d\omega/dt$.

Note. The above relation, in terms of angular acceleration may also be written as :

$$\alpha = \frac{a}{r} \quad \textbf{Ans.}$$

Example 21.11. *A car is moving at 72 k.m.p.h., If the wheels are 75 cm diameter, find the angular velocity of the tyre about its axis. If the car comes to rest in a distance of 20 metres, under a uniform retardation, find angular retardation of the wheels.*

Solution. Given : Linear velocity (v) = 72 k.m.p.h. = 20 m/s; Diameter of wheel (d) = 75 cm or radius (r) = 37.5 m = 0.375 m and distance travelled by the car (s) = 20 m.

Angular retardation of the wheel

We know that the angular velocity of the wheel,

$$\omega = \frac{v}{r} = \frac{20}{0.375} = 53.3 \text{ rad/sec}$$

Let a = Linear retardation of the wheel.

We know that $v^2 = u^2 + 2as$

$\therefore$ $0 = (20)^2 + 2 \times a \times 20 = 400 + 40a$

or $a = -\frac{400}{40} = -10 \text{ m/sec}^2$ ...(Minus sign indicates retardation)

We also know that the angular retardation of the wheel,

$$\alpha = \frac{a}{r} = \frac{-10}{0.375} = -26.7 \text{ rad/sec}^2 \quad \textbf{Ans.}$$

...(Minus sign indicates retardation)

Motion of rotation of a body under variable angular acceleration:

Example 21.12. *The equation for angular displacement of a body moving on a circular path is given by :*

$$\theta = 2t^3 + 0.5$$

where θ is in rad and t in sec. Find angular velocity, displacement and acceleration after 2 sec.

Solution. Given : Equation for angular displacement $\theta = 2t^3 + 0.5$ ...(*i*)

Angular displacement after 2 seconds

Substituting t = 2 in equation (*i*),

$$\theta = 2\,(2)^3 + 0.5 = 16.5 \text{ rad} \quad \textbf{Ans.}$$

Angular velocity after 2 seconds

Differentiating both sides equation (*i*) with respect to t,

$$\frac{d\theta}{dt} = 6\,t^2 \qquad \text{...(ii)}$$

or velocity, $\omega = 6\,t^2$ ...(*iii*)

Substituting t = 2 in equation (*iii*),

$$\omega = 6\,(2)^2 = 24 \text{ rad/sec} \quad \textbf{Ans.}$$

Angular acceleration after 2 seconds

Differentiating both sides of equation (*iii*) with respect to t,

$$\frac{d\omega}{dt} = 12t \text{ or Acceleration } \alpha = 12t$$

Now substituting t = 2 in above equation,

$$\alpha = 12 \times 2 = 24 \text{ rad/sec}^2 \quad \textbf{Ans.}$$

Example [redacted]. *The equation for angular displacement of a particle, moving in a circular path (radius 200 m) is given by :*

$$\theta = 18t + 3t^2 - 2t^3$$

where θ is the angular displacement at the end of t sec. Find (i) angular velocity and acceleration at start, (ii) time when the particle reaches its maximum angular velocity; and (iii) maximum angular velocity of the particle.

Solution. Given : Equation for angular displacement $\theta = 18t + 3t^2 - 2t^3$ ...(*i*)

(*i*) *Angular velocity and acceleration at start*

Differentiating both sides of equation (*i*) with respect to *t*,

$$\frac{d\theta}{dt} = 18 + 6t - 6t^2$$

i.e. angular velocity, $\omega = 18 + 6t - 6t^2$ ...(*ii*)

Substituting $t = 0$ in equation (*ii*),

$$\omega = 18 + 0 - 0 = 18 \text{ rad/s} \quad \textbf{Ans.}$$

Differentiating both sides of equation (*ii*) with respect to *t*,

$$\frac{d\omega}{dt} = 6 - 12t$$

i.e. angular acceleration, $\alpha = 6 - 12t$ ...(*iii*)

Now substituting $t = 0$ in equation (*iii*),

$$\alpha = 6 \text{ rad/s}^2 \quad \textbf{Ans.}$$

(*ii*) *Time when the particle reaches maximum angular velocity*

For maximum angular velocity, differentiating the equation for angular velocity (*ii*) with respect to *t i.e.* equation (*iii*) and equating it to zero.

$$6 - 12t = 0 \quad \text{or} \quad t = \frac{6}{12} = 0.5 \text{ sec} \quad \textbf{Ans.}$$

(*iii*) *Maximum angular velocity of the particle*

The maximum angular velocity of the particle may now be found out by substituting $t = 0.5$ in equation (*ii*),

$$\omega_{max} = 18 + (6 \times 0.5) - 6\,(0.5)^2 = 19.5 \text{ rad/s} \quad \textbf{Ans.}$$

OBJECTIVE TYPE QUESTIONS

1. The angular velocity of rotating body is expressed in terms of
   (*a*) revolution per minute (*b*) radians per second
   (*c*) any one of the two (*d*) none of the two
2. The linear velocity of a rotating body is given by the relation
   (*a*) $v = r.\omega$ (*b*) $v = r/\omega$
   (*c*) $v = \omega/r$ (*d*) ω^2/r
   where r = Radius of the circular path, and
   ω = Angular velocity of the body in radians/s.
3. The linear acceleration of a rotating body is given by the relation
   (*a*) $a = r.\alpha$ (*b*) $a = r/\alpha$
   (*c*) $a = \alpha/r$ (*d*) α^2/r
   where r = Radius of the circular path, and
   α = Angular acceleration of the body in radians/s^2
4. If at any given instant, we know that linear velocity and acceleration of a car, we can mathematically obtain its
   (*a*) angular velocity (*b*) angular acceleration
   (*c*) none of the two (*d*) both of the two
5. The relationship between linear velocity and angular velocity of a cycle
   (*a*) exists under all conditions
   (*b*) does not exist under all conditions
   (*c*) exists only when it does not slip
   (*d*) exists only when it moves on horizontal plane

ANSWERS

1. (*c*) **2.** (*a*) **3.** (*a*) **4.** (*d*) **5.** (*a*)